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Spectral fluctuation properties of constrained unitary ensembles of Gaussian-distributed random matrices

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Abstract
We investigate the spectral fluctuation properties of constrained ensembles of random matrices (defined by the condition that a number N_Q of matrix elements vanish identically; that condition is imposed in unitarily invariant form) in the limit of large matrix dimension. We show that as long as N_Q is smaller than a critical value (at which the quadratic level repulsion of the Gaussian unitary ensemble of random matrices may be destroyed) all spectral fluctuation measures have the same form as for the Gaussian unitary ensemble.

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1. Introduction

We investigate the spectral fluctuation properties of the constrained unitary ensembles of Gaussian-distributed random matrices (CGUE) introduced in [1]. Constrained ensembles of random matrices deserve interest because they represent entire classes of non-canonical random-matrix ensembles that were proposed to mimic typical properties of interacting many-fermion systems more closely than do the canonical ensembles (the Gaussian orthogonal, unitary and symplectic ensembles) [2]. Examples of constrained ensembles are the embedded Gaussian orthogonal ensemble [3, 4] and the two-body random ensemble [5–7]. The constraints essentially require certain matrix elements or linear combinations of matrix elements to vanish. It is difficult to deal with the constrained ensembles analytically because they lack the invariance properties that give analytical access to the canonical ensembles. That difficulty is overcome by imposing the condition of unitary, orthogonal or symplectic invariance on the constrained ensembles. It is expected that that condition leaves the spectral fluctuations unchanged (whereas the eigenfunctions acquire the same distribution as for the canonical ensembles). Here we focus on the case of unitary invariance, i.e., on the CGUE.

Some spectral properties of the CGUE were exhibited in [1]. In particular, the following sufficient condition for level repulsion was given. The quadratic level repulsion characteristic of the Gaussian unitary ensemble (GUE) (the ensemble of Gaussian-distributed unitary random matrices) also prevails for the CGUE provided that the number N_Q of constraints does not exceed a critical value,

$$N_Q < N_Q^{\text{crit}} \tag{1}$$

with N_Q^{crit} defined in equation (16).

In the present paper, we go beyond [1] and address the spectral fluctuation properties of the CGUE in the limit of large matrix dimension $N \gg 1$. We do so for $N_Q < N_Q^{\text{crit}}$. For $N_Q = 0$ the CGUE coincides with the GUE. For $N_Q \geqslant N_Q^{\text{crit}}$ the form of the constraints does not seem to permit definitive analytical statements. It has remained an open question to what extent the spectral fluctuation properties of the CGUE (beyond the statement of sheer level repulsion) are the same or differ from those of the GUE for $0 < N_Q < N_Q^{\text{crit}}$. We prove the following.

Theorem. *For matrix dimension $N \gg 1$ and the number of constraints $N_Q < N_Q^{\text{crit}}$ (with N_Q^{crit} defined in equation (16)), the spectral fluctuation measures of the CGUE coincide with those of the GUE saved for correction terms of order $1/N$.*

To make the paper self-contained, we collect in section 2 some definitions and results given in [1]. In section 3, we slightly modify the definition of the constrained ensembles so as to remove a singularity. In section 4, we discuss the form of the constraints in the limit $N \gg 1$. Our proof is given in section 5. It is based on an approach developed in [8]. Section 6 contains a discussion. Some technical details are presented in the appendix.

2. Definitions

We consider Hermitean matrices acting on a Hilbert space $\mathcal{H}$ of dimension N. For any two such matrices A and B, we introduce the canonical scalar product in terms of the trace

$$\langle A|B\rangle \equiv \text{Tr}(AB). \tag{2}$$

This allows us to define an orthonormal basis of N^2 Hermitean basis matrices $B_\alpha = B_\alpha^\dagger$ which obey

$$\langle B_\alpha|B_\beta\rangle \equiv \text{Tr}(B_\alpha B_\beta) = \delta_{\alpha\beta} \tag{3}$$

and

$$\sum_{\alpha=1}^{N^2} |B_\alpha\rangle\langle B_\alpha| = \mathbf{1}_N, \tag{4}$$

where $\mathbf{1}_N$ is the unit matrix in N dimensions. Any Hermitean matrix H acting on $\mathcal{H}$ can be expanded in terms of the N^2 Hermitean basis matrices B_α as

$$H = \sum_{\alpha=1}^{N^2} h_\alpha B_\alpha. \tag{5}$$

The Gaussian unitary ensemble of random matrices is obtained by assuming that the expansion coefficients h_α are uncorrelated Gaussian-distributed real random variables with mean value zero and a common variance. For the GUE, the probability density $W(H)$ of the matrix elements of H has the form

$$W(H)\,\mathrm{d}[H] = \mathcal{N}^{\text{GUE}} \exp\left(-\frac{N}{2\lambda^2}\langle H|H\rangle\right)\mathrm{d}[H]. \tag{6}$$

Here $\mathrm{d}[H]$ is the product of the differentials of all independent matrix elements, $\mathcal{N}^{\mathrm{GUE}}$ is a normalization factor and 2λ is the radius of Wigner's semicircle.

We introduce the constraints by considering two orthogonal subspaces labeled $\mathcal{P}$ and $\mathcal{Q}$ with dimensions N_P and $N_Q = N^2 - N_P$, respectively. These are defined in terms of orthogonal projection operators

$$\mathcal{P} = \sum_{p=1}^{N_P} |B_p\rangle\langle B_p|, \qquad \mathcal{Q} = \sum_{q=N_P+1}^{N^2} |B_q\rangle\langle B_q|. \tag{7}$$

We have

$$\begin{aligned} &\mathcal{P}^\dagger = \mathcal{P}, \qquad \mathcal{Q}^\dagger = \mathcal{Q}, \qquad \mathcal{P}^2 = \mathcal{P}, \\ &\mathcal{Q}^2 = \mathcal{Q}, \qquad \mathcal{P}\mathcal{Q} = 0, \qquad \mathcal{P} + \mathcal{Q} = \mathbf{1}_N. \end{aligned} \tag{8}$$

Constraints can be formulated in the form $\langle H|\mathcal{Q}\rangle = 0$ for all H. In the CGUE such constraints are used in unitarily invariant form. The CGUE is defined by the probability density $W_{\mathcal{P}}$ of the matrix elements of H given by

$$W_{\mathcal{P}}(H)\,\mathrm{d}[H] = \mathcal{N}^{\mathrm{GUE}} \exp\left(-\frac{N}{2\lambda^2}\langle H|H\rangle\right)\mathrm{d}[H]\int \mathrm{d}[U]\left(\prod_q \delta\left(\sqrt{\frac{N}{2\pi\lambda^2}}\langle U B_q U^\dagger|H\rangle\right)\right). \tag{9}$$

The integral $\mathrm{d}[U]$ extends over the unitary group in N dimensions. The Haar measure of the unitary group is normalized to one, i.e.,

$$\int \mathrm{d}[U] = 1. \tag{10}$$

We diagonalize the matrix H with the help of a unitary matrix V,

$$H = V x V^\dagger, \tag{11}$$

where $x = \mathrm{diag}(x_1, \ldots, x_N)$ is the diagonal matrix of the eigenvalues. The integration measure becomes

$$\mathrm{d}[H] \propto \Delta^2(x)\,\mathrm{d}[x]\mathrm{d}[V], \tag{12}$$

where $\mathrm{d}x$ is the product of the differentials of the N eigenvalues, where $\mathrm{d}[V]$ is the Haar measure of the unitary group in N dimensions, and where $\Delta(x)$ denotes the Vandermonde determinant

$$\Delta(x) = \prod_{1\leqslant\mu<\nu\leqslant N} (x_\mu - x_\nu). \tag{13}$$

Equation (12) shows that eigenvalues and eigenvectors of the CGUE are uncorrelated random variables. The joint probability distribution $P_{\mathcal{P}}(x)$ of the eigenvalues is given by

$$P_{\mathcal{P}}(x) = \mathcal{N}_0 \exp\left(-\frac{N}{2\lambda^2}\langle x|x\rangle\right)\Delta^2(x) F_{\mathcal{P}}(H), \tag{14}$$

where

$$\begin{aligned} F_{\mathcal{P}}(H) &\equiv \int \mathrm{d}[U]\left(\prod_q \delta\left(\sqrt{\frac{N}{2\pi\lambda^2}}\langle B_q|U H U^\dagger\rangle\right)\right) \\ &= \int \mathrm{d}[U]\left(\prod_q \delta\left(\sqrt{\frac{N}{2\pi\lambda^2}}\langle B_q|U x U^\dagger\rangle\right)\right) \end{aligned} \tag{15}$$

is a function of the eigenvalues $\{x_\mu\}$ only, and where $\mathcal{N}_0$ is another irrelevant normalization factor. Comparison of equations (6) and (9) shows that the eigenvalue distribution of the CGUE differs from that of the GUE by the factor $F_{\mathcal{P}}(H)$. GUE-type level repulsion is contained in the factor $\Delta^2(x)$ in equation (14), and such level repulsion will prevail also in the CGUE unless $F_{\mathcal{P}}(H)$ is singular whenever two eigenvalues coincide. In [1] it was shown that $F_{\mathcal{P}}(H)$ cannot be singular if the number N_Q of constraints obeys the inequality

$$N_Q < N_Q^{\text{crit}} = N(N-1)/2 - \sum_{j=1}^{J} L_j(L_j - 1)/2. \tag{16}$$

Here it is assumed that the matrix $B = \sum_q s_q B_q$ with real coefficients s_q possesses asymptotically (all s_q large) J sets of degenerate eigenvalues with multiplicities L_j, $j = 1, \ldots, J$.

3. Modified form of the constraints

The function $F_{\mathcal{P}}(H)$ embodies the constraints. Therefore, it is the central object of study in this paper. The treatment of $F_{\mathcal{P}}(H)$ simplifies when all confining matrices B_q are traceless. We believe that that case is physically the more interesting one, for the following reason. We show in the appendix that whenever the B_q s are not traceless, there always exists an orthogonal transformation of the set $\{B_q\}$ to a new set $\{\tilde{B}_q\}$ such that $F_{\mathcal{P}}(H)$ is unchanged and that all $\tilde{B}_q$ with $q > N_P + 1$ are traceless. The one constraining matrix $\tilde{B}_{N_P+1}$, that is not traceless, is the sum of a traceless part and of a multiple of the unit matrix. But constraining H with a multiple of the unit matrix means that we constrain the centroid of the spectrum of H. We cannot think of a physically interesting situation where such a constraint would be meaningful. This is why we focus attention on the case where all B_q are traceless,

$$\langle B_q \rangle = 0 \qquad \text{for all} \quad q = N_P + 1, N_P + 2, \ldots, N^2, \tag{17}$$

and treat the more general case where condition (17) is violated, in the appendix. Here and in the following we use the symbol $\langle A \rangle$ to denote the trace of the matrix A. This is consistent with definition (2).

For the developments in section 5 we note the following properties of $F_{\mathcal{P}}(H)$. The function $F_{\mathcal{P}}(H)$ is real (this follows from equation (15)) and positive definite (this is seen when we write the defining delta functions as limits of Gaussians). Using Fourier transformation, we can write $F_{\mathcal{P}}(H)$ in equation (15) as an N_Q-fold Fourier integral,

$$F_{\mathcal{P}}(H) = \left(\frac{\lambda^2}{2\pi N}\right)^{N_Q/2} \prod_{q=1}^{N_Q} \int \mathrm{d}s_q \int \mathrm{d}[U] \exp\{\mathrm{i}\langle B(s)|UHU^\dagger\rangle\}, \tag{18}$$

where

$$B(s) = \sum_q s_q B_q \tag{19}$$

and where s stands for the set $\{s_1, \ldots, s_{N_Q}\}$. The integral over the unitary group can be worked out and with $\mathrm{d}s = \prod_q \mathrm{d}s_q$ yields (see [1])

$$F_{\mathcal{P}}(H) \propto \left(\frac{\lambda^2}{2\pi N}\right)^{N_Q/2} \int \mathrm{d}s \frac{\det \exp\{\mathrm{i}x_\mu b_\nu(s)\}}{\Delta(x)\Delta(b(s))}. \tag{20}$$

Here the $b_\nu(s)$ are the eigenvalues of the matrix $B(s)$, and $\Delta(b)$ is the Vandermonde determinant of the $b_\nu(s)$, see equation (13). In [1] it was shown that the integrals over s

converge if condition (16) is met. Moreover, inspection of equation (20) shows that $F_{\mathcal{P}}(H)$ is not singular when two eigenvalues x_μ, x_ν coincide.

However, because of the form of the constraints (equation (15)), the function $F_{\mathcal{P}}(H)$ is singular when all eigenvalues of H coincide. To see this we define

$$\tilde{H} = H - \frac{\mathbf{1}_N}{N}\langle H\rangle, \tag{21}$$

use assumption (17) and rewrite equation (18) in the form

$$F_{\mathcal{P}}(H) = F_{\mathcal{P}}(\tilde{H}) = \left(\frac{\lambda^2}{2\pi N}\right)^{N_Q/2} \int \mathrm{d}s \int \mathrm{d}[U] \exp\{\mathrm{i}\langle B(s)|U\tilde{H}U^\dagger\rangle\}. \tag{22}$$

Equation (22) shows that $F_{\mathcal{P}}(\tilde{H})$ is singular when $\tilde{H} = 0$, i.e., when all eigenvalues of H coincide. The singularity mirrors a singularity in definition (15) of $F_{\mathcal{P}}(H)$. Indeed, when $x_\mu = x_\nu = y$ for all $\mu, \nu = 1, \ldots, N$, the Dirac deltas in definition (15) take the form $\delta(y\langle B_q\rangle)$. Because of equation (17) each of these terms is singular. We avoid the singularity by modifying the definition of $F_{\mathcal{P}}(\tilde{H})$. Instead of $F_{\mathcal{P}}(\tilde{H})$, we consider the constraining function

$$\tilde{F}_{\mathcal{P}}(\tilde{H}) = \left(\frac{\langle\tilde{H}^2\rangle}{N\lambda^2}\right)^{N_Q/2} F_{\mathcal{P}}(\tilde{H}). \tag{23}$$

The factor in front of $F_{\mathcal{P}}(\tilde{H})$ guarantees that $\tilde{F}_{\mathcal{P}}(\tilde{H})$ is not singular at $\tilde{H} = 0$. At the same time, that factor is a function of the sum of the eigenvalues x_μ only. Thus, that factor cannot modify the correlations of close-lying eigenvalues x_μ, and the spectral fluctuation properties of the constrained ensembles defined by the constraining functions $F_{\mathcal{P}}(\tilde{H})$ and $\tilde{F}_{\mathcal{P}}(\tilde{H})$ are the same. Moreover, for real x_μ the function $\tilde{F}_{\mathcal{P}}(\tilde{H})$ is real and positive definite. According to equation (20), $\tilde{F}_{\mathcal{P}}(\tilde{H})$ is not singular for finite values of the x_μ. Inspection shows that when one of the eigenvalues, x_μ say, tends to infinity, $\tilde{F}_{\mathcal{P}}(\tilde{H})$ cannot grow more strongly than some power of x_μ. That growth is much weaker than the Gaussian suppression of large eigenvalues in equation (9). Hence, the confinement of the spectrum to a finite interval characteristic of the GUE persists also for the CGUE with constraining function $\tilde{F}_{\mathcal{P}}(\tilde{H})$ although the shape of the average spectrum may be modified.

Collecting everything, we have

$$\tilde{F}_{\mathcal{P}}(\tilde{H}) = \left(\frac{\langle(\tilde{H})^2\rangle}{2\pi N^2}\right)^{N_Q/2} \int \mathrm{d}[U] \int \mathrm{d}s \exp\{\mathrm{i}\langle U\,B(s)U^\dagger|\tilde{H}\rangle\}. \tag{24}$$

It is convenient to introduce the new variables $t_q = \lambda s_q$. Then

$$\tilde{F}_{\mathcal{P}}(\tilde{H}) = \left(\frac{\langle(\tilde{H})^2\rangle}{2\pi\lambda^2 N^2}\right)^{N_Q/2} \int \mathrm{d}[U] \int \mathrm{d}t \exp\{\mathrm{i}\langle U\,B(t)U^\dagger|(\tilde{H}/\lambda)\rangle\}. \tag{25}$$

This shows that $\tilde{F}_{\mathcal{P}}(\tilde{H})$ depends on $\tilde{H}$ only via the dimensionless ratio $\tilde{H}/\lambda$, as expected. Because of unitary invariance, $\tilde{F}_{\mathcal{P}}(\tilde{H})$ can depend only on unitary invariants constructed from $\tilde{H}/\lambda$. The only such invariants are the normalized traces of $\tilde{H}^n/\lambda^n$ with positive integer n. For $N \gg 1$ this is shown explicitly in the following section. The probability density for the Hamiltonian matrices of the CGUE is given by

$$\tilde{W}_{\mathcal{P}}(H)\,\mathrm{d}[H] = \tilde{\mathcal{N}} \exp\left(-\frac{N}{2\lambda^2}\langle H|H\rangle\right)\tilde{F}_{\mathcal{P}}(\tilde{H})\,\mathrm{d}[H]. \tag{26}$$

The substitution of $F_{\mathcal{P}}(\tilde{H})$ by $\tilde{F}_{\mathcal{P}}(\tilde{H})$ also modifies the normalization factor of $\tilde{W}_{\mathcal{P}}$ but that is irrelevant for what follows.

4. Asymptotic form of $\tilde{F}_{\mathcal{P}}(\tilde{H})$ for $N \gg 1$

For $N \gg 1$ we now display explicitly the dependence of $\tilde{F}_{\mathcal{P}}(\tilde{H})$ on the normalized unitary invariants $(1/N)\langle \tilde{H}^n/\lambda^n\rangle$ with positive integer n. We mention in passing that the terms of leading order in a systematic expansion of $\tilde{F}_{\mathcal{P}}(\tilde{H})$ in powers of N_Q/N^2 can also be obtained from the Harish–Chandra–Itzykson–Zuber integral [9, 10], or from the standard supersymmetry approach [11, 12]. We use assumption (17) and discuss the case where not all constraining matrices are traceless in the appendix. Then $\langle B(t)\rangle = 0$. We consider the expressions

$$\int \mathrm{d}[U](\mathrm{i}\langle(\tilde{H}/\lambda)|UB(t)U^\dagger\rangle)^k \tag{27}$$

with k positive integer. Such expressions are generated when the exponential in equation (25) is expanded in a Taylor series. To calculate the integral over the unitary group we use a method valid for $N \gg 1$ [13, 14]. To leading order in $1/N$ the integral can be done using Wick contraction on the matrices U, the rules being

$$U_{\mu\nu}U^\dagger_{\rho\sigma} \to (1/N)\delta_{\mu\sigma}\delta_{\nu\rho} \qquad \text{and} \qquad U_{\mu\nu}U_{\rho\sigma} \to 0. \tag{28}$$

Terms of higher order are obtained in a similar fashion and lead to similar results but are not considered here. We first look at a few simple cases. For $k = 1$ expression (27) vanishes. For $k = 2$ and $k = 3$ we obtain $(\mathrm{i}^2/N)[(1/N)\langle(\tilde{H}/\lambda)^2\rangle]\langle B^2(t)\rangle$ and $2!(\mathrm{i}^3/N)^2[(1/N)\langle(\tilde{H}/\lambda)^3\rangle]\langle B^3(t)\rangle$, respectively. For $k = 4$ Wick contraction generates two terms. One is proportional to the square of the $k = 2$ term just considered. The other is given by $3!(\mathrm{i}^4/N^3)[(1/N)\langle(\tilde{H}/\lambda)^4\rangle]\langle B^4(t)\rangle$. For $k = 5$, Wick contraction generates two types of terms: the product of the $k = 2$ term and the $k = 3$ term, and a new term given by $4!(\mathrm{i}^5/N^4)[(1/N)\langle(\tilde{H}/\lambda)^5\rangle]\langle B^5(t)\rangle$.

For the general expression (27) we consider all partitions of k into sets of positive integers $k_1, k_2, \ldots, k_f$ greater than unity such that $\sum_{i=1}^f k_i = k$. To leading order in $1/N$, expression (27) is given by the sum over all such partitions, the contribution of each partition being $\mathrm{i}^k\binom{k}{k_1}\binom{k-k_1}{k_2} \times \cdots \times \binom{k-k_1-\cdots-k_{f-1}}{k_f}\prod_{i=1}^f (k_i-1)!(1/N^{k_i-1})[(1/N)\langle(\tilde{H}/\lambda)^{k_i}\rangle]\langle B^{k_i}(t)\rangle$. The terms of higher order in $1/N$ also involve products of traces of powers of $\tilde{H}/\lambda$ and of traces of B, the difference being that at least one trace of a power of $\tilde{H}/\lambda$ is multiplied by at least two traces of powers of B such that the sum of the exponents of B equals the exponent of $\tilde{H}/\lambda$. It is shown below that $\langle B^n\rangle$ and $\prod_i\langle B^{n_i}\rangle$ with $\sum_i n_i = n$ are of the same order in N so the neglect of such terms is legitimate.

We conclude that to leading order in $1/N$, the integral over the unitary group in equation (25) is given by

$$\int \mathrm{d}[U]\exp\{\mathrm{i}\langle UB(t)U^\dagger|(\tilde{H}/\lambda)\rangle\} = \exp\left\{\sum_{n\geqslant 2}(1/n)(\mathrm{i}^n/N^{n-1})[(1/N)\langle(\tilde{H}/\lambda)^n\rangle]\langle B^n(t)\rangle\right\}. \tag{29}$$

For $N \gg 1$ the constraining function $\tilde{F}$ is then given by

$$\tilde{F}_{\mathcal{P}}(\tilde{H}) = \left(\frac{\langle(\tilde{H})^2\rangle}{2\pi\lambda^2N^2}\right)^{N_Q/2}\int \mathrm{d}t\,\exp\left\{\sum_{n\geqslant 2}(1/n)(\mathrm{i}^n/N^{n-1})[(1/N)\langle(\tilde{H}/\lambda)^n\rangle]\langle B^n(t)\rangle\right\}. \tag{30}$$

It may seem that because of the factors N^{n-1} the terms of higher order in n in equations (29) and (30) can be neglected. We now show that for $N_Q \sim N^2$ this is not the case.

In equation (30), we replace the Cartesian integration variables t_q by polar coordinates $\{r, \Omega\}$ in N_Q dimensions where

$$r^2 = \sum_q t_q^2 \tag{31}$$

and where Ω stands for the angular variables. We write

$$B(t) = r\,B(\Omega) \tag{32}$$

and since $\langle B_q | B_{q'} \rangle = \delta_{qq'}$ have

$$\langle B^2(\Omega) \rangle = 1. \tag{33}$$

Let $b_\mu(\Omega)$ denote the N real eigenvalues of $B(\Omega)$. Then $\sum_\mu b_\mu^2(\Omega) = 1$ and $|b_\mu(\Omega)| \leqslant 1$ for all $\mu = 1, \ldots, N$. For integer $n > 2$ this implies that

$$\langle B^n(\Omega) \rangle \leqslant 1. \tag{34}$$

This, incidentally, justifies the omission of terms of order $1/N$ above and shows that $\langle B^n(\Omega) \rangle$ and $\langle (\tilde{H}/\lambda)^n \rangle$ are characteristically different: the first expression is (at most) of order unity while the second is of order N. That is why we always carry the second expression in the form $(1/N)\langle (\tilde{H}/\lambda)^n \rangle$.

Using the transformation to polar coordinates we observe that the term with $n = 2$ in equation (30) gives a Gaussian integral in r. Expanding the terms with $n > 2$ in the exponent in a Taylor series we are led to consider radial integrals of the form

$$\int \mathrm{d}r\, r^{N_Q - 1 + 2k} \exp\{-cr^2\}. \tag{35}$$

Here c is a constant and $2k$ must be even as otherwise the integrals vanish. Compared to the leading term ($k = 0$) these integrals yield a factor $(N_Q + 2k - 2)(N_Q + 2k - 4) \times \cdots \times N_Q$. (We assume for simplicity that N_Q is even). If the expansion of the exponential converges sufficiently rapidly so that for $N_Q \sim N^2$ we need consider only terms with $k \ll N_Q$ then every power of r in the exponential in equation (30) effectively carries a factor $\sqrt{N_Q}$, and the series in n proceeds effectively in powers of N_Q/N^2. For $N_Q \sim N^2$ that factor is of order unity.

While it is, thus, not permitted to terminate for $N_Q \sim N^2$ the series in n in equation (30) with the first few terms, rapid convergence of the Taylor expansion of the exponential around the Gaussian form is assured by the following property of the matrix $B(\Omega)$ defined in equation (32). Each term in the Taylor expansion of the right-hand side of equation (30) around the Gaussian form ($n = 2$) generates a factor of the form

$$\int \mathrm{d}\Omega \prod_i \langle B^{k_i}(\Omega) \rangle, \qquad \text{where} \quad \sum_i k_i = \text{even} = 2k \text{ and where all } k_i \geqslant 3. \tag{36}$$

That expression can also be written as

$$\int \mathrm{d}\Omega \prod_i \sum_{\mu=1}^{N} b_\mu^{k_i}(\Omega). \tag{37}$$

In magnitude, each eigenvalue $b_\mu(\Omega)$ is bounded by unity. It is, therefore, safe to expect that on averaging over the N_Q-dimensional unit sphere, each eigenvalue is of order $1/\sqrt{N}$ so that the expression in equation (37) is of order $N^{-k}\Omega(N_Q)$. Here $\Omega(N_Q)$ is the surface of the unit sphere in N_Q dimensions. That shows that only few terms in the expansion are expected to contribute significantly even for $N_Q \sim N^2$. The statement holds *a fortiori* for $N_Q \ll N^2$.

5. Proof

To calculate the influence of the constraints in equation (26) on the spectrum for $N \gg 1$, we use the approach developed in [8] based on the supersymmetry method [11, 12]. We only sketch the essential steps, using the definitions and notation of [8, 12]. The average level density and all correlation functions are obtained with the help of a generating functional Z which is written as

$$Z = \int \mathrm{d}\Psi \left\langle \exp\left\{\frac{\mathrm{i}}{2}\Psi^\dagger \mathbf{L}^{1/2}\mathbf{G}\mathbf{L}^{1/2}\Psi\right\}\right\rangle_H, \tag{38}$$

where

$$\mathbf{G} = \mathbf{H} - \mathbf{E} + \mathbf{M}. \tag{39}$$

The average over the ensemble is denoted by the angular brackets with an index H while for the trace we continue to use angular brackets without index as in equation (2). The symbol Ψ stands for a supervector the dimension of which depends on the particular correlation function under study. The same is true of the matrices $\mathbf{H}$ (the Hamiltonian), $\mathbf{E}$ (the energy) and $\mathbf{M}$. The matrix $\mathbf{M}$ is of order $O(N^{-1})$ and contains energy differences, source terms and possible couplings to channels. Differentiation with respect to the source terms generates the particular correlation function of interest.

The invariance of CGUE under unitary transformations implies that the integrand in equation (38) depends upon Ψ and $\Psi^\dagger$ only via the invariant form

$$A_{\alpha\beta} = N^{-1} L^{1/2}_{\alpha\alpha} \sum_{\mu=1}^{N} \Psi_{\mu\alpha}\Psi^\dagger_{\mu\beta} L^{1/2}_{\beta\beta}. \tag{40}$$

Here α and β are matrix indices in superspace while μ runs over the N basis states of Hilbert space $\mathcal{H}$. We introduce a supermatrix σ with the same dimension and symmetry properties as A by writing Z as an integral over a delta function,

$$Z = \int \mathrm{d}\Psi \int \mathrm{d}\sigma\, \delta(\sigma - A) \left\langle \exp\left\{\frac{\mathrm{i}}{2}\Psi^\dagger \mathbf{L}^{1/2}\mathbf{G}\mathbf{L}^{1/2}\Psi\right\}\right\rangle_H. \tag{41}$$

The delta function is replaced by its Fourier transform, and the multiple Gaussian integral over the supervector Ψ is performed to yield

$$Z = \int \mathrm{d}\sigma \int \mathrm{d}\tau \exp\left\{\frac{\mathrm{i}}{2} N \operatorname{trg}(\tau\sigma)\right\} \left\langle \exp\left\{-\frac{1}{2}\operatorname{tr}\operatorname{trg}\ln[\mathbf{G} - \lambda\tau]\right\}\right\rangle_H. \tag{42}$$

The remaining superintegrations over τ and σ are eventually done for $N \to \infty$ with the help of the saddle-point approximation. Prior to that step, the average over the ensemble is performed using equation (26). We first integrate over the unitary group. To leading order in N^{-1} we obtain

$$\left\langle \exp\left\{-\frac{1}{2}\operatorname{tr}\operatorname{trg}\ln[\mathbf{G} - \lambda\tau]\right\}\right\rangle_H = \left\langle \exp\left\{-\frac{1}{2}\operatorname{tr}\operatorname{trg}\ln \mathbf{D}\right\} \times \exp\left\{-\frac{1}{2}\operatorname{tr}\operatorname{trg}\ln\left[1 + \frac{1}{N}\operatorname{tr} D^{-1}\mathbf{M}\right]\right\}\right\rangle_H. \tag{43}$$

Here $\mathbf{D} = \mathbf{x} - \mathbf{E} - \lambda\tau$ is a diagonal in Hilbert space, and the angular brackets now stand for the remaining integration over the eigenvalues $\{x_\mu\}$. Under inclusion of the terms which arise from $\tilde{W}_\mathcal{P}$ in equation (26), the exponent of the integrand is given by

$$-\frac{1}{2}\operatorname{tr}\operatorname{trg}\ln \mathbf{D} - \frac{1}{2}\operatorname{tr}\operatorname{trg}\ln\left[1 + \frac{1}{N}\operatorname{tr}\mathbf{D}^{-1}\mathbf{M}\right] + 2\sum_{\mu<\nu}\ln|x_\mu - x_\nu| - \frac{N}{2\lambda^2}\sum_\mu x_\mu^2 + \ln\tilde{F}_\mathcal{P}(\tilde{H}). \tag{44}$$

Following [8], we perform the eigenvalue integration using the saddle-point approximation for $N \gg 1$. In expression (44), all terms in the first line are at most of order N while the first two terms in the second line are of order N^2. Omitting the terms in the first line but keeping $\ln \tilde{F}_{\mathcal{P}}(\tilde{H})$ (depending on the value of N_Q, that function may or may not be of order N^2), we put the derivatives of the resulting expression with respect to the x_μ equal to zero and obtain the N saddle-point equations

$$x_\mu = \frac{2\lambda^2}{N} \sum_{\sigma \neq \mu} \frac{1}{x_\mu - x_\sigma} + \frac{2\lambda^2}{N} \frac{1}{\tilde{F}_{\mathcal{P}}(\tilde{x})} \frac{\partial \tilde{F}_{\mathcal{P}}(\tilde{H})}{\partial x_\mu}, \qquad \mu = 1, \ldots, N. \tag{45}$$

Without the term $\ln \tilde{F}_{\mathcal{P}}(\tilde{H})$ in expression (44), the saddle-point equations would have taken the standard GUE form

$$x_\mu = \frac{2\lambda^2}{N} \sum_{\sigma \neq \mu} \frac{1}{x_\mu - x_\sigma}. \tag{46}$$

To prepare for the treatment of equations (45) we recall how equations (46) are solved in [15]. The variables x_μ/λ are replaced by a single dimensionless continuous variable, $x_\mu/\lambda \to \varepsilon$. The normalized level density $\rho(\varepsilon)$ with $\int \mathrm{d}\varepsilon\, \rho(\varepsilon) = 1$ of the ensemble (which at this point is an unknown function) is introduced, and equations (46) are written in the form

$$\varepsilon = 2P \int \mathrm{d}\varepsilon' \frac{\rho(\varepsilon')}{\varepsilon - \varepsilon'}. \tag{47}$$

Here $P \int$ stands for the principal-value integral. Equation (47) has an electrostatic analogue and can be solved using the theory of analytic functions. The result is Wigner's semicirle law for $\rho(\varepsilon)$. The generating functional is subsequently taken at the saddle-point values for the x_μ. All summations over x_μ in Z are, thus, replaced by integrations over ε with $\rho(\varepsilon)$ as weight function.

In applying that same method to equations (45) we introduce the (yet unknown) normalized average level density $\rho_{\mathcal{P}}(\varepsilon)$ of the constrained ensemble in the sum on the right-hand side of equations (45). We also have to implement the change of variables $x_\mu/\lambda \to \varepsilon$ in $\tilde{F}_{\mathcal{P}}(\tilde{H})$ and its derivative. As for $\tilde{F}_{\mathcal{P}}(\tilde{H})$, this is done by replacing everywhere in equation (30) the term $(1/N)\langle (H/\lambda)^n \rangle$ by $\langle \varepsilon^n \rangle = \int \mathrm{d}\varepsilon\, \varepsilon^n \rho_{\mathcal{P}}(\varepsilon)$. The form of $\tilde{W}(H)$ in equation (26) implies that $\rho_{\mathcal{P}}(\varepsilon) = \rho_{\mathcal{P}}(-\varepsilon)$ so that only terms with n even survive, and we obtain

$$\tilde{F}_{\mathcal{P}} = \left(\frac{\langle \varepsilon^2 \rangle}{2\pi N} \right)^{N_Q/2} \int \mathrm{d}t \exp \left\{ \sum_{n \geqslant 1} (1/(2n))((-1)^n/N^{2n-1}) \langle \varepsilon^{2n} \rangle \langle B^{2n}(t) \rangle \right\}. \tag{48}$$

This is a function of the unknown level density $\rho_{\mathcal{P}}$ with a rapidly converging Taylor expansion around the Gaussian term ($n = 1$). In the derivative of $\tilde{F}_{\mathcal{P}}$, we substitute $x_\mu/\lambda \to \varepsilon$ after differentiating with respect to x_μ. We obtain

$$\frac{\lambda}{N} \frac{\partial \tilde{F}_{\mathcal{P}}}{\partial x_\mu} = \frac{N_Q \varepsilon}{N^2 \langle \varepsilon^2 \rangle} \tilde{F}_{\mathcal{P}} + \left(\frac{\langle \varepsilon^2 \rangle}{2\pi N} \right)^{N_Q/2} \sum_{n \geqslant 1} \frac{(-)^n \varepsilon^{2n-1}}{N^{2n+1}} \int \mathrm{d}t \langle B^{2n}(t) \rangle$$
$$\times \exp \left\{ \sum_{n \geqslant 1} (1/(2n))((-1)^n/N^{2n-1}) \langle \varepsilon^{2n} \rangle \langle B^{2n}(t) \rangle \right\}. \tag{49}$$

The right-hand side of equation (49) is a polynomial of odd order in ε with rapidly decreasing coefficients.

As a result, the saddle-point equations (45) take the form of equation (47), with ε replaced by an odd-order polynomial in ε with rapidly decreasing coefficients. These coefficients

depend on $\rho_{\mathcal{P}}(\varepsilon)$; the solution to equation (49) must, therefore, proceed iteratively, with the GUE average level density as a starting point for calculating $\langle \varepsilon^n \rangle$. In [1] it was shown perturbatively that $\rho(E)$ and $\rho_{\mathcal{P}}(E)$ differ by a term of order N_Q/N^2. Therefore, we expect the two level densities to differ significantly when $N_Q \sim N^2$.

Returning to equation (44) we perform the integration over the variables x_μ by taking their values at the saddle points. That means, for instance, that we write

$$\frac{\lambda}{N} \sum_{\mu} \frac{1}{x_\mu - E - \lambda\tau} \to \int \mathrm{d}E' \frac{\rho_{\mathcal{P}}(E'/\lambda)}{E' - E - \lambda\tau}. \tag{50}$$

This is the essential step: the summations over the eigenvalues x_μ disappear in all expressions in the integrand of Z. Each such summation is replaced by an energy integral involving the level density $\rho_{\mathcal{P}}(\varepsilon)$ of the constrained ensemble. This is the only place where the constraints show up in the calculation. From here on the calculation of the correlation functions for the CGUE and that for the GUE run completely in parallel [8]. It follows that all correlation functions of the CGUE have the same form as their GUE counterparts except that we have to replace the local average level spacing of the GUE by that of the CGUE. This proves our theorem.

6. Discussion

Our theorem holds in the limit $N \gg 1$ and for $N_Q < N_Q^{\mathrm{crit}}$. The average level density of the CGUE may differ from that of the GUE but all correlation functions have the same form for both ensembles. Although our result is perhaps expected, to the best of our knowledge this is the first time that GUE-type statistics has been analytically proved for a class of ensembles different from the GUE. Deviations of order $1/N$ from the asymptotic form of the GUE statistics exist, of course, even for the pure GUE and are expected *a fortiori* for the CGUE. Our proof specifically applies to the case of unitary invariance. We believe, however, that a corresponding result holds also for the other symmetries.

The proof of the theorem rests on the fact that in the limit $N \gg 1$ and for $N_Q < N_Q^{\mathrm{crit}}$, the constraining function $\tilde{F}_{\mathrm{P}}(\tilde{H})$ is free of singularities. The proof holds independently of any specific properties of the constraining matrices B_q. What happens for $N \gg 1$ but $N_Q \geqslant N_Q^{\mathrm{crit}}$? That seems to depend on specific properties of the constraints which determine the eigenvalues $b_\mu(s)$ and, thus, the convergence properties of the integrals over s. Therefore, generic statements about the spectral fluctuation properties of the CGUE probably cannot be made for $N_Q \geqslant N_Q^{\mathrm{crit}}$.

One may speculate that with N_Q increasing beyond the value N_Q^{crit}, the spectral fluctuations of the CGUE remain GUE-like until $N_P = N^2 - N_Q$ is reduced to the value $N_P = N$ (where $H_P = \sum_p h_p B_p$ may be a linear combination of N commuting matrices and, thus, integrable). But that speculation is surely incorrect. Indeed, random band matrices with a band width less than or of order $\sqrt{N}$ are known [16, 17] to possess localized eigenfunctions and a Poisson spectrum. For such matrices, the number N_Q of constraints is at least of order $N^2 - N\sqrt{N}$ and, for $N \gg 1$, obviously much larger than N_Q but still much smaller than $N^2 - N$.

It is of interest to discuss the embedded random k-body ensembles EGUE(k) (see [3] and the review [4]) in the light of these considerations. The EGUE(k) models a Fermionic many-body system with k-body interactions: m identical spinless Fermions occupy l degenerate single-particle states. The Hilbert space is spanned by $N = \binom{l}{m}$ Slater determinants. To construct the k-body interaction operators, we denote by $a_\mu^\dagger$ and a_μ the creation and destruction operators for a Fermion in the single-particle state labeled μ with $\mu = 1, \ldots, l$. Let $\mu_1, \ldots, \mu_m$ with $1 \leqslant \mu_i \leqslant l$ for all i denote a set of m non-equal integers, and analogously

for $\nu_1, \ldots, \nu_m$ with $1 \leqslant \nu_j \leqslant l$ for all j. Then a general interaction operator has the form $A(\{\mu_i\}, \{\nu_j\}) = \prod_{i=1}^{m} a_{\mu_i}^{\dagger} \prod_{j=1}^{m} a_{\nu_j}$. In the Hilbert space of Slater determinants, the N^2 Hermitean operators $A(\{\mu_i\}, \{\nu_j\}) + A^{\dagger}(\{\mu_i\}, \{\nu_j\})$ and $\mathrm{i}[A(\{\mu_i\}, \{\nu_j\}) - A^{\dagger}(\{\mu_i\}, \{\nu_j\})]$ play the very same role as do the matrices B_α introduced in section 2 in the Hilbert space $\mathcal{H}$. From the general form of A, a k-body operator is obtained by imposing the condition that a subset of $m - k$ elements of the set $\{\mu_i\}$ is identically equal to a subset of $m - k$ elements of the set $\{\nu_j\}$. There are $\binom{l}{m}\binom{m}{k}\binom{l-m}{k}$ such k-body operators. The EGUE(k) is obtained by writing the Hamiltonian as a linear combination of all k'-body operators with $k' \leqslant k$ and with coefficients that are real random Gaussian-distributed variables.

Among the EGUE(k), the EGUE(2) has received particular attention because it mimics a Hamiltonian with two-body interactions, a form typical for Fermionic many-body systems like atoms or nuclei. One of the central questions (undecided so far) has been whether for $N \gg 1$ the spectral fluctuation properties of EGUE(2) are GUE-like. Numerical simulations [4] suggest that the answer is affirmative. However, these simulations are typically done for small values of l and m, with l around 12 and m around 4 or so. But for these values the number of one- plus two-body interaction terms is $\binom{12}{4}[4 \times 8 + 6 \times 28] = 200\binom{12}{4}$. The number of constraints is accordingly given by $N_Q = \binom{12}{4}[495 - 200] = \binom{12}{4} \times 295$. That figure is not much larger than $N_Q^{\mathrm{crit}} = \binom{12}{4} \times 247$, so that it is difficult to draw firm conclusions. It would be more informative to investigate numerically large values of l and m but that is prohibitively difficult. For $l \gg m \gg 1$ we have $N \approx l^m$, and the number of independent two-body operators is approximately $l^m m^2 l^2$. In other words, there are only $m^2 l^2$ non-zero interaction matrix elements in every row and column of the matrix representation of the Hamiltonian for the EGUE(2), much fewer than for a banded random matrix where that number would be approximately $\sqrt{N} = l^{m/2}$. Put differently, the number N_Q of constraints for the EGUE(2) is much bigger than it is for a banded random matrix. That fact suggests that mixing in the EGUE(2) is weaker than it is for a banded random matrix, and that EGUE(2) has Poissonian level statistics. On the other hand, in a banded random matrix it takes approximately $\sqrt{N}$ different interaction matrix elements to connect two arbitrary states in Hilbert space. In the EGUE(2) that number is only $m/2$, i.e., less even than $(1/2) \ln N$. This fact suggests that mixing of the states in Hilbert space is much more efficient for the EGUE(2) than it is for a banded random matrix, and the question remains undecided. But the discussion suggests that for $N_Q \geqslant N_Q^{\mathrm{crit}}$, the form of the constraints (and not just their sheer number) becomes important in determining the spectral fluctuation properties of CGUE.

Another frequently used ensemble that simulates the nuclear many-body system is the two-body random ensemble (TBRE), see [5, 6] and the review [7]. Actually that ensemble is taken to be invariant under time reversal and, thus, has orthogonal rather than unitary symmetry. For simplicity we disregard this fact. The single-particle states belonging to a major shell of the nuclear shell model are occupied by a number of nucleons. The resulting Slater determinants are coupled to states with fixed total spin J and isospin T and are written as $|JT\mu\rangle$. The running index μ has a typical range R from several ten (J large) to several thousand or more (J small). Level statistics can be meaningfully discussed only for large R. It is assumed that the interaction between nucleons is of two-body type. Within a major shell, the number of independent two-body matrix elements v_α is small (of order 10 or 10^2) compared to the large values of R that are of interest. These matrix elements are taken to be uncorrelated Gaussian-distributed random variables. This defines the TBRE. For a given set of states $|JT\mu\rangle$, the matrix representation of the Hamiltonian H_{TB} of the TBRE takes the form

$$(H_{\mathrm{TB}})_{\mu\nu} = \sum_{\alpha} v_\alpha C_{\mu\nu}^{JT}(\alpha). \tag{51}$$

The matrices $C^{JT}_{\mu\nu}(\alpha)$ are fixed by the major shell and by the quantum numbers J and T under consideration but have some properties in common with matrices drawn from a canonical random-matrix ensemble. Again, it is of interest whether in the limit of $R \gg 1$ the TBRE generically obeys GUE (or GOE) level statistics. Numerical results and semi-analytical arguments both support such a hypothesis. Unfortunately, the matrices $C^{JT}_{\mu\nu}(\alpha)$ are not accessible analytically so far. Therefore, it does not even seem possible to determine the number N_Q of constraints that would characterize the TBRE matrix (51), and we cannot apply our results to that ensemble.

The authors of [1] considered not only the CGUE but in addition also what they called 'deformed Gaussian ensembles'. Here the delta functions in equation (15) are replaced by Gaussians, and the constraining function $F_{\mathcal{P}}(H)$ is everywhere regular. Following the arguments in section 5 we conclude that the spectral fluctuations of the deformed ensembles coincide with those of the GUE. In other words, constraints affect the spectral fluctuation properties only if they constrain the relevant matrix elements to the value zero (and not to very small non-zero values). As a consequence, in the GUE the transition from GUE to Poisson level statistics is not a continuous process (where the level statistics would be smoothly deformed) but actually happens discontinuously. These statements apply on the 'macroscopic' level where the values of the coefficients h_q of the constraining matrices B_q are compared with those of the h_p multiplying B_p. If, on the other hand, the h_q are measured in units of the mean level spacing (i.e., in effect, on a scale $1/\sqrt{N}$ compared to the scale of the h_p) then the transition from GUE to Poisson level statistics is expected to be smooth and to allow for intermediate forms of the level statistics. That expectation is supported by transitions between symmetries like the GOE $\to$ GUE transition, and by many examples of partially chaotic systems that show intermediate level statistics. We have not attempted to introduce a correspondingly scaled parametrization for the deformed ensembles. Such a step would be meaningful only in the immediate vicinity of the transition point from GUE to Poisson level statistics. That point is not known analytically, however.

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Appendix. Constraining matrices with non-zero traces

It is convenient to relabel the indices q so that they run from 1 to N_Q. We use equation (18) and rotate the basis $B_q \to \tilde{B}_q = \sum_{q'} O_{qq'} B_{q'}$ with the help of an orthogonal transformation $O_{qq'}$ such that $\tilde{B}_1$ points in the direction of the unit matrix $\mathbf{1}_N$. Then,

$$\begin{aligned} &\langle \tilde{B}_q | \tilde{B}_{q'} \rangle = \delta_{qq'} \qquad \text{for all} \quad q, q', \\ &\langle \tilde{B}_q \rangle = 0 \qquad\qquad\ \ \text{for all} \quad q > 1. \end{aligned} \tag{A.1}$$

We apply the same orthogonal transformation to the variables t_q so that $t_q \to \tilde{t}_q = \sum_{q'} O_{qq'} t_{q'}$ and obtain

$$F_{\mathcal{P}}(H) = \left(\frac{\lambda^2}{2\pi N}\right)^{N_Q/2} \int \prod_q \mathrm{d}\tilde{t}_q \int \mathrm{d}[U] \exp\left(\mathrm{i} \sum_q \tilde{t}_q \langle \tilde{B}_q | U H U^\dagger \rangle\right). \tag{A.2}$$

We write $\tilde{B}_1$ as the sum of a traceless part and of a multiple of the unit matrix,

$$\tilde{B}_1 = \left(\tilde{B}_1 - \langle\tilde{B}_1\rangle\frac{\mathbf{1}_N}{N}\right) + \langle\tilde{B}_1\rangle\frac{\mathbf{1}_N}{N}. \tag{A.3}$$

By construction, the traceless part is orthogonal to all the $\tilde{B}_q$'s with $q > 1$ and has norm

$$\left\langle \tilde{B}_1 - \langle\tilde{B}_1\rangle\frac{\mathbf{1}_N}{N} | \tilde{B}_1 - \langle\tilde{B}_1\rangle\frac{\mathbf{1}_N}{N}\right\rangle = 1 - \frac{1}{N}\langle\tilde{B}_1\rangle^2 = \alpha^2. \tag{A.4}$$

Because of the first of equations (A.1), the eigenvalues $b_{1\mu}$ with $\mu = 1, \ldots, N$ of $\tilde{B}_1$ obey $\sum_\mu b_{1\mu}^2 = 1$. We maximize $\langle\tilde{B}_1\rangle = \sum_\mu b_{1\mu}$ under that constraint and find that

$$-\sqrt{N} \leqslant \langle\tilde{B}_1\rangle \leqslant \sqrt{N}. \tag{A.5}$$

Therefore,

$$0 \leqslant \alpha^2 \leqslant 1. \tag{A.6}$$

We define α as the positive root of α^2 and write equation (A.3) in the form

$$\tilde{B}_1 = \alpha\hat{B}_1 + \langle\tilde{B}_1\rangle\frac{\mathbf{1}_N}{N}. \tag{A.7}$$

Then, $\hat{B}_1$ has trace zero and norm one. We define $\hat{B}_q = \tilde{B}_q$ for all $q > 1$ and have

$$\langle\hat{B}_q\rangle = 0 \qquad \text{for all} \quad q \qquad \text{and} \qquad \langle\hat{B}_q|\hat{B}_{q'}\rangle = \delta_{qq'} \qquad \text{for all} \quad q, q'. \tag{A.8}$$

For $\alpha = 0$ the matrix $\tilde{B}_1$ is a multiple of the unit matrix, and the integral over $\tilde{t}_1$ in equation (A.2) yields a multiple of the delta function for $\langle H\rangle$ while the remaining $N_Q - 1$ integrations over the $\tilde{t}_q$ with $q > 1$ are treated as in section 4. For $\alpha = 1$ the matrix $\tilde{B}_1$ is actually traceless; that case was treated in section 4. Therefore, we consider α only in the open interval

$$0 < \alpha < 1. \tag{A.9}$$

We rewrite equation (A.2) by using decomposition (A.7), by rescaling $\alpha\tilde{t}_1 \to \tilde{t}_1$, by introducing spherical polar coordinates $\{t, \Omega\}$ in N_Q dimensions, and by defining the matrix $B(\Omega)$ as

$$t\,B(\Omega) = \sum_q \tilde{t}_q\hat{B}_q. \tag{A.10}$$

The function $F_{\mathcal{P}}(H)$ takes the form

$$F_{\mathcal{P}}(H) = \left(\frac{\lambda^2}{2\pi N}\right)^{N_Q/2}\frac{1}{\alpha}\int \mathrm{d}t\, t^{N_Q-1}\,\mathrm{d}\Omega\exp(\mathrm{i}\tilde{t}_1(t,\Omega)\langle\tilde{B}_1\rangle\langle H\rangle/(\alpha N))$$
$$\times\int \mathrm{d}[U]\exp(\mathrm{i}t\langle B(\Omega)|UHU^\dagger\rangle). \tag{A.11}$$

Here $\tilde{t}_1(t, \Omega)$ stands for the rescaled old integration variable $\tilde{t}_1$ as expressed in terms of the new integration variables $\{t, \Omega\}$. The function $F_{\mathcal{P}}(H)$ diverges for $\langle H\rangle \to 0$. The divergence is removed by multiplying $F_{\mathcal{P}}(H)$ with $[\langle H/\lambda\rangle^2]^{N_Q/2}$. As in the case of the substitution $F_{\mathcal{P}} \to \tilde{F}_{\mathcal{P}}$ in section 3, we expect that this step does not affect the spectral fluctuations of the ensemble. The matrix $B(\Omega)$ is traceless by construction, and the integration over the unitary group can be carried out as in section 4. From here on we proceed as in section 5.

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